

CASTELNUOVO-MUMFORD REGULARITY OF SEMINORMAL SIMPLICIAL AFFINE SEMIGROUP RINGS

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ABSTRACT. We show that the Eisenbud-Goto conjecture holds for (homogeneous) seminormal simplicial affine semigroup rings. Moreover, we prove an upper bound for the Castelnuovo-Mumford regularity in terms of the dimension, which is similar as in the normal case. Finally, we compute explicitly the regularity of full Veronese rings.

1. INTRODUCTION

Let K be a field, and let $R = K[x_1, \dots, x_n]$ be a standard graded polynomial ring, that is, all variables x_i have degree 1. Let M be a finitely generated graded R -module. By $H_{R_+}^i(M)$ we denote the i -th local cohomology module of M with respect to the homogeneous maximal ideal R_+ of R , and we set $a(H_{R_+}^i(M)) := \max \{r \mid H_{R_+}^i(M)_r \neq 0\}$ with the convention $a(0) = -\infty$. The *Castelnuovo-Mumford regularity* (or *regularity* for short) $\text{reg } M$ of M is defined by

$$\text{reg } M := \max \{i + a(H_{R_+}^i(M)) \mid i \geq 0\}.$$

The regularity $\text{reg } M$ is an important invariant, for example, the i -th syzygy module of M can be generated by elements of degree smaller or equal to $\text{reg } M + i$, moreover, one can use the regularity of a homogeneous ideal to bound the degrees in certain minimal Gröbner bases; for more information we refer to the paper of Eisenbud and Goto [EG84], and to Bayer and Stillman [BS87]. So it is natural to ask for bounds for the regularity of a homogeneous ideal I of R ; note that $\text{reg } I = \text{reg } R/I + 1$. Denote by $\text{codim } R/I := \dim_K[R/I]_1 - \dim R/I$ the codimension of R/I and by $\deg R/I$ its degree. An open conjecture is

Conjecture 1.1 (Eisenbud-Goto [EG84]). *If K is algebraically closed and I is a homogeneous prime ideal of R , then*

$$\text{reg } R/I \leq \deg R/I - \text{codim } R/I.$$

By a result of Gruson, Lazarsfeld, and Peskine [GLP83] Conjecture 1.1 holds if $\dim R/I = 2$. The Cohen-Macaulay case was proven by Treger [Tre82], and the Buchsbaum case by Stückrad and Vogel [SV87]. Conjecture 1.1 also holds if $\deg R/I \leq \text{codim } R/I + 2$ by a result of Hoa, Stückrad, and Vogel [HSV91], and in characteristic zero for smooth surfaces by Lazarsfeld [Laz87] and for certain smooth threefolds by Ran [Ran90]. Moreover, Giaimo [Gia06] showed that the conjecture still holds for connected reduced curves.

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Since the Eisenbud-Goto conjecture is widely open, it would be nice to prove it for more cases; in the following we will consider homogeneous simplicial affine semigroup rings. A semigroup is called *affine* if it is finitely generated and isomorphic to a submonoid of $(\mathbb{Z}^m, +)$ for some $m \in \mathbb{N}^+$. Let B be an affine semigroup. The *affine semigroup ring* $K[B]$ associated to B is defined as the K -vector space with basis $\{t^b \mid b \in B\}$ and multiplication given by the K -bilinear extension of $t^a \cdot t^b = t^{a+b}$. Since B is an affine semigroup we have $G(B) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}$; where $G(B)$ denotes the group generated by B . Hence $G(B) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite dimensional \mathbb{R} -vector space with canonical embedding $G(B) \subseteq G(B) \otimes_{\mathbb{Z}} \mathbb{R}$ given by $x \mapsto x \otimes 1$. We say that B is *simplicial* if the corresponding cone $C(B)$ is generated by linearly independent elements, where $C(X) := \{\sum_{i=1}^k r_i x_i \mid k \in \mathbb{N}^+, r_i \in \mathbb{R}_{\geq 0}, x_i \in X\}$ for $X \subseteq G(B) \otimes_{\mathbb{Z}} \mathbb{R}$. An element $x \in B$ is called a *unit* if $-x \in B$. We say that B is *positive* if 0 is its only unit. In this case, the *Hilbert basis* $\text{Hilb}(B)$, that is, the set of irreducible elements of B , is a unique minimal generating set of B ; an element $x \in B$ is called *irreducible* if it is not a unit and if for $x = y + z$ with $y, z \in B$ it follows that y or z is a unit. Moreover, we say that B is *homogeneous* if B is positive and there is a positive \mathbb{Z} -grading on $K[B]$ in which every t^b for $b \in \text{Hilb}(B)$ has degree 1. See [BG09, Chapter 2]. In the following we will assume that B is homogeneous. We will always consider the above \mathbb{Z} -grading on $K[B]$, moreover, by $\text{reg } K[B]$ we mean the regularity of $K[B]$ with respect to the canonical R -module structure which is induced by the homogeneous surjective K -algebra homomorphism

$$\pi : R = K[x_1, \dots, x_n] \twoheadrightarrow K[B],$$

given by $x_i \mapsto t^{a_i}$; where $\text{Hilb}(B) = \{a_1, \dots, a_n\}$. Hence $R/\ker \pi \cong K[B]$, where $\ker \pi$ is a homogeneous prime ideal of R . In case that B is simplicial we will also call $K[B]$ a simplicial affine semigroup ring.

By extending the ground field if necessary, (the inequality in) Conjecture 1.1 holds for $K[B]$ in particular if $\dim K[B] = 2$, if $K[B]$ is Buchsbaum, and if $\deg K[B] \leq \text{codim } K[B] + 2$. The conjecture also holds if $\text{codim } K[B] = 2$ by Peeva and Sturmfels [PS98] and for simplicial affine semigroup rings with isolated singularity by Herzog and Hibi [HH03]. In [HS03, Theorem 3.2], Hoa and Stückrad presented a very good bound for the regularity of simplicial affine semigroup rings, moreover, they provided some cases where Conjecture 1.1 holds in the simplicial case. However, the Eisenbud-Goto conjecture is still widely open even for simplicial affine semigroup rings. In case that B is simplicial and seminormal (see Definition 3.1) we can confirm the Eisenbud-Goto conjecture for $K[B]$, we obtain the following

Main Result (Theorem 3.14, Theorem 3.18). *Let K be an arbitrary field and let B be a homogeneous affine semigroup. If B is simplicial and seminormal, then*

$$\text{reg } K[B] \leq \min \{ \dim K[B] - 1, \deg K[B] - \text{codim } K[B] \}.$$

This result is more or less well known if B is normal (see Definition 3.1), since $K[B]$ is Cohen-Macaulay in this case; see [Hoc72, Theorem 1], [BG09, Theorem 6.10], and Remark 3.10. In fact, the ring $K[B]$ is not necessary Buchsbaum if B is simplicial and seminormal, see Example 3.5. To prove Conjecture 1.1 in the seminormal simplicial case we will use an idea of Hoa and Stückrad and decompose the ring $K[B]$ into a direct sum of certain monomial ideals. This becomes even more powerful in this case, since seminormality of simplicial affine semigroup rings can be characterized in terms of the decomposition by a result of Li [Li04].

In Section 2 we will recall the decomposition of simplicial affine semigroup rings. Moreover, we will introduce sequences with $*$ -property which will be useful to prove the main result in Section 3. Finally, we will compute explicitly the Castelnuovo-Mumford regularity of full Veronese rings in Section 4. We set $M_{d,\alpha} := \{(u_1, \dots, u_d) \in \mathbb{N}^d \mid \sum_{i=1}^d u_i = \alpha\}$ where $d, \alpha \in \mathbb{N}^+$, moreover, we define $B_{d,\alpha}$ to be the submonoid of $(\mathbb{N}^d, +)$ which is generated by $M_{d,\alpha}$. In Theorem 4.2 we will show that $\text{reg } K[B_{d,\alpha}] = \lfloor d - \frac{d}{\alpha} \rfloor$. For a general consideration of seminormal rings we refer to [Swa80, Tra70], and for unspecified notation to [BG09, Eis95].

2. BASICS

In the following we will assume that the homogeneous affine semigroup B is simplicial, that is, there are linearly independent elements $e_1, \dots, e_d \in C(B)$ such that $C(B) = C(\{e_1, \dots, e_d\})$. Without loss of generality we may assume that $e_1, \dots, e_d \in \text{Hilb}(B)$. Consider the \mathbb{R} -vector space isomorphism $\varphi : \text{span}(\{e_1, \dots, e_d\}) \rightarrow \mathbb{R}^d$ where e_i is mapped to the element in \mathbb{N}^d all of whose coordinates are zero except the i -th coordinate which is equal to α for some $\alpha \in \mathbb{N}^+$, that is, $\varphi(e_i) = (0, \dots, 0, \alpha, 0, \dots, 0)$. By construction we have $\varphi(B) \subseteq \mathbb{R}_{\geq 0}^d$, since $C(B) = C(\{e_1, \dots, e_d\})$, hence $\varphi(B) \subseteq \mathbb{Q}_{\geq 0}^d$ by the Gaussian elimination. Thus, by choosing a suitable α we may assume that $\varphi(\text{Hilb}(B)) \subset \mathbb{N}^d$, or equivalently, $\varphi(B) \subseteq \mathbb{N}^d$. The affine semigroup $\varphi(B)$ is again homogeneous, it follows that the coordinate sum of all elements of $\varphi(\text{Hilb}(B))$ is equal to α , see [BG09, Proposition 2.20]. Note that we can compute $\text{reg } K[B]$ in terms of $H_{K[B]_+}^i(K[B])$, since $H_{R_+}^i(K[B]) \cong H_{K[B]_+}^i(K[B])$, see [BS98, Theorem 13.1.6]; where $K[B]_+$ denotes the homogeneous maximal ideal of $K[B]$. The isomorphism $B \cong \varphi(B)$ of semigroups induces an isomorphism of \mathbb{Z} -graded rings $K[B] \cong K[\varphi(B)]$. This enables us to identify a homogeneous simplicial affine semigroup B with its image $\varphi(B)$ in \mathbb{N}^d . Thus, we may assume that B is the submonoid of $(\mathbb{N}^d, +)$ which is generated by a set $\{e_1, \dots, e_d, a_1, \dots, a_c\} \subseteq M_{d,\alpha}$, where

$$e_1 := (\alpha, 0, \dots, 0), e_2 := (0, \alpha, 0, \dots, 0), \dots, e_d := (0, \dots, 0, \alpha).$$

Let $a_i = (a_{i[1]}, \dots, a_{i[d]})$; since $\alpha \in \mathbb{N}^+$ can be chosen to be minimal, we may assume that the integers $a_{i[j]}$, $i = 1, \dots, c$, $j = 1, \dots, d$, are relatively prime. Moreover, we assume that $c \geq 1$, since the case $c = 0$ is not relevant in our context. Note that K is an arbitrary field, $\dim K[B] = d$, and $\text{codim } K[B] = c$. Our notation tries to follow the notation in [HS03].

By $x_{[i]}$ we denote the i -th component of x and $\deg x := (\sum_{j=1}^d x_{[j]})/\alpha$, for $x \in G(B)$. We define $A := \langle e_1, \dots, e_d \rangle$ to be the submonoid of B generated by e_1, \dots, e_d , and we set

$$B_A := \{x \in B \mid x - a \notin B \ \forall a \in A \setminus \{0\}\}.$$

Note that B_A is finite. Moreover, if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$. We define $x \sim y$ if $x - y \in G(A) = \alpha\mathbb{Z}^d$, thus, \sim is an equivalence relation on $G(B)$. Every element of $G(B)$ is equivalent to an element of $G(B) \cap D$, where $D := \{(x_{[1]}, \dots, x_{[d]}) \in \mathbb{Q}^d \mid 0 \leq x_{[i]} < \alpha \ \forall i\}$ and for all $x, y \in G(B) \cap D$ with $x \neq y$ we have $x \not\sim y$. Hence the number of equivalence classes $f := \#(G(B) \cap D)$ on $G(B)$ is finite, moreover, there are also f equivalence classes on B and on B_A . By $\Gamma_1, \dots, \Gamma_f$ we denote the equivalence classes on B_A . For $t = 1, \dots, f$ we define

$$h_t := (\min \{m_{[1]} \mid m \in \Gamma_t\}, \min \{m_{[2]} \mid m \in \Gamma_t\}, \dots, \min \{m_{[d]} \mid m \in \Gamma_t\}).$$

Note that $x - h_t \in A$ for all $x \in \Gamma_t$. This shows that $h_t \in G(B) \cap \mathbb{N}^d$. Let $T := K[y_1, \dots, y_d]$ be a standard graded polynomial ring, that is, all variables y_i have degree 1. We define

$\tilde{\Gamma}_t := \{y^{(x-h_t)/\alpha} \mid x \in \Gamma_t\}$, where $u/\alpha := (u_{[1]}/\alpha, \dots, u_{[d]}/\alpha)$ and $y^u := y_1^{u_{[1]}} \cdot \dots \cdot y_d^{u_{[d]}}$ for $u = (u_{[1]}, \dots, u_{[d]}) \in \mathbb{N}^d$. We obtain $\tilde{\Gamma}_t \subset T$, and therefore $I_t := \tilde{\Gamma}_t T$ is a monomial ideal in T for all $t = 1, \dots, f$. It follows that $\text{ht } I_t \geq 2$ (height), since $\gcd \tilde{\Gamma}_t = 1$. By T_+ we denote the homogeneous maximal ideal of T . See [HS03, Section 2]. We have:

Proposition 2.1 ([HS03, Proposition 2.2]). *There are isomorphisms of \mathbb{Z} -graded T -modules:*

- (1) $K[B] \cong \bigoplus_{t=1}^f I_t(-\deg h_t)$.
- (2) $H_{K[B]_+}^i(K[B]) \cong \bigoplus_{t=1}^f H_{T_+}^i(I_t)(-\deg h_t)$ for all $i \geq 0$.

It follows that $\deg K[B] = f$. Moreover, we have

$$(2.1) \quad \text{reg } K[B] = \max \{\text{reg } I_t + \deg h_t \mid t = 1, \dots, f\},$$

where $\text{reg } I_t$ denotes the regularity of I_t as a \mathbb{Z} -graded T -module. This shows that the regularity of $K[B]$ is independent of K for $\dim K[B] \leq 5$ by [BH97, Corollary 1.4].

Remark. This decomposition can be computed by using the MACAULAY2 [GS] package MONOMIALALGEBRAS [BEN], which has been developed by Janko Böhm, David Eisenbud, and the author. In this package we consider the case of affine semigroups $Q' \subseteq Q \subseteq \mathbb{N}^d$ such that $K[Q]$ is finite over $K[Q']$; the implemented algorithm decomposes the ring $K[Q]$ into a direct sum of monomial ideals in $K[Q']$. There is also an algorithm implemented computing $\text{reg } K[Q]$ in the homogeneous case, moreover, there are functions available testing the Buchsbaum, Cohen-Macaulay, Gorenstein, normal, and the seminormal property in the simplicial case. Note that this decomposition works more general, for more information we refer to [BEN11].

Definition 2.2. For an element $x \in B$ we say that a sequence $\lambda = (b_1, \dots, b_n)$ has **-property* if $b_1, \dots, b_n \in \{e_1, \dots, e_d, a_1, \dots, a_c\}$ and $x - b_1 \in B, x - b_1 - b_2 \in B, \dots, x - (\sum_{j=1}^n b_j) \in B$; we say that the *length* of λ is n . Let $\lambda = (b_1, \dots, b_n)$ be a sequence with *-property of x ; we define $x(\lambda, i) := x - (\sum_{j=1}^i b_j)$ for $i = 1, \dots, n$, and $x(\lambda, 0) := x$. By Λ_x we denote the set of all sequences with *-property of x with length $\deg x$, with the convention $\Lambda_0 := \emptyset$.

By construction we have $\Lambda_x \neq \emptyset$ for all $x \in B \setminus \{0\}$. The definition of a sequence with *-property is motivated to control the degree of $K[B]$, the second assertion in Lemma 2.4 illustrates the usefulness of this construction. For elements $x, y \in G(B)$ we define $x \geq y$ if $x_{[k]} \geq y_{[k]}$ for all $k = 1, \dots, d$.

Remark 2.3. Let $\lambda = (b_1, \dots, b_n)$ be a sequence with *-property of x . We get $x(\lambda, i) \geq x(\lambda, j)$ for $0 \leq i \leq j \leq n$. Moreover, we have $\deg x(\lambda, i) = \deg x - i$ for $i = 0, \dots, n$. Hence for $\lambda \in \Lambda_x$ we get $x(\lambda, \deg x) = 0$.

Lemma 2.4. *Let $x \in B_A \setminus \{0\}$ and $\lambda = (b_1, \dots, b_n)$ be a sequence with *-property of x . Then*

- (1) $x(\lambda, i) \in B_A$ for all $i = 0, \dots, n$.
- (2) $x(\lambda, i) \not\sim x(\lambda, j)$ for all $i, j \in \mathbb{N}$ with $0 \leq i < j \leq n$.

Proof. (1) Follows from construction since if $y \notin B_A$ then $y + z \notin B_A$ for all $y, z \in B$.

(2) Suppose to the contrary that $x(\lambda, i) \sim x(\lambda, j)$ for some $i, j \in \mathbb{N}$ with $0 \leq i < j \leq n$. We have $x(\lambda, i) \geq x(\lambda, j)$, hence

$$x(\lambda, i) = x(\lambda, j) + \sum_{t=1}^d n_t e_t$$

for some $n_t \in \mathbb{N}$. Since $\deg x(\lambda, i) > \deg x(\lambda, j)$ we get that $n_t > 0$ for some $t \in \{1, \dots, d\}$. Thus, $x(\lambda, i) - e_t \in B$ and therefore $x(\lambda, i) \notin B_A$ which contradicts claim (1). \square

Remark 2.5. Let $x \in B_A \setminus \{0\}$ and $\lambda = (b_1, \dots, b_n)$ be a sequence with $*$ -property of x . Suppose that $b_j \in \{e_1, \dots, e_d\}$ for some $j \in \{1, \dots, n\}$. Hence $x - b_j = x(\lambda, n) + \sum_{k=1, k \neq j}^n b_k \in B$ which contradicts $x \in B_A$. This shows that $b_1, \dots, b_n \in \{a_1, \dots, a_c\}$.

Lemma 2.4 implies that $\deg x \leq \deg K[B] - 1$ for all $x \in B_A$. This bound can be improved by using the following observation:

Remark 2.6. Consider the set $L = \{0, a_1, \dots, a_c\}$, by construction $L \subseteq B_A$. Let $x \in L$ and $y \in B_A$ with $x \neq y$; suppose that $x \sim y$. Since $0 \leq x_{[i]} < \alpha$ for all $i = 1, \dots, d$, we have $y \geq x$. By a similar argument as in Lemma 2.4 (2) we get $y \notin B_A$. This shows that $x \not\sim y$.

Proposition 2.7 ([HS03, Theorem 1.1]). *We have $\deg x \leq \deg K[B] - \text{codim } K[B]$ for all $x \in B_A$.*

Proof. Let $x \in B_A \setminus \{0\}$, and $\lambda \in \Lambda_x$. By Lemma 2.4 and Remark 2.6 we get a set

$$L := \{0, a_1, \dots, a_c\} \cup \{x(\lambda, 0), \dots, x(\lambda, \deg x - 1)\},$$

with $L \subseteq B_A$ such that $y \not\sim z$ for all $y, z \in L$ with $y \neq z$. Hence

$$\deg K[B] = f \geq \#L = \deg x + \text{codim } K[B].$$

□

We note that this proof is a new proof of [HS03, Theorem 1.1]. We define the *reduction number* $r(K[B])$ of $K[B]$ by $r(K[B]) := \max \{\deg x \mid x \in B_A\}$, see [HS03, Page 129, 135]. It follows that

$$(2.2) \quad r(K[B]) \leq \deg K[B] - \text{codim } K[B],$$

that is, the Eisenbud-Goto conjecture holds for the reduction number of $K[B]$. So whenever we have $\text{reg } K[B] = r(K[B])$ the Eisenbud-Goto conjecture holds. It should be mentioned that this property does not hold in general. Even for a monomial curve in \mathbb{P}^3 the equality does not hold. For $B = \langle (40, 0), (0, 40), (35, 5), (11, 29) \rangle$ we get $\text{reg } K[B] = 13 > 11 = r(K[B])$. Note that we always have $r(K[B]) \leq \text{reg } K[B]$ by Equation (2.1).

Example 2.8. Consider the monoid $B = \langle (4, 0), (0, 4), (3, 1), (1, 3) \rangle$. We have

$$B_A = \{(0, 0), (3, 1), (1, 3), (6, 2), (2, 6)\},$$

and therefore $r(K[B]) = \max \{0, 1, 1, 2, 2\} = 2$. We get

$$\Gamma_1 = \{(0, 0)\}, \Gamma_2 = \{(3, 1)\}, \Gamma_3 = \{(1, 3)\}, \Gamma_4 = \{(6, 2), (2, 6)\},$$

and $h_1 = (0, 0), h_2 = (3, 1), h_3 = (1, 3), h_4 = (2, 2)$. By this we have $I_1 = I_2 = I_3 = T$ and $I_4 = (y_1, y_2)T$, hence

$$\text{reg } K[B] = \max \{\text{reg } T + 0, \text{reg } T + 1, \text{reg } T + 1, \text{reg}(y_1, y_2)T + 1\} = \max \{0, 1, 1, 2\} = 2.$$

Lemma 2.9. *Let $x \in B_A, t \in \mathbb{N}^+, q \in \{1, \dots, d\}$, and $x_{[q]} = t\alpha$. There exists a $\lambda \in \Lambda_x$ such that $(t-1)\alpha < x(\lambda, 1)_{[q]} < t\alpha$.*

Proof. Fix a $\nu = (b_1, \dots, b_{\deg x}) \in \Lambda_x$. We have $x(\nu, \deg x) = 0$ by Remark 2.3, hence there is a $k \in \{1, \dots, \deg x\}$ with $b_{k[q]} > 0$. Since $b_k \in \{a_1, \dots, a_c\}$ by Remark 2.5 we get that $b_{k[q]} < \alpha$. The claim follows from the fact that $(b_{\sigma(1)}, \dots, b_{\sigma(\deg x)}) \in \Lambda_x$ for every permutation σ of $\{1, \dots, \deg x\}$, since $x = \sum_{j=1}^{\deg x} b_j$. □

The next combinatorial Lemma will be useful to prove the Eisenbud-Goto conjecture in the seminormal case in Theorem 3.18.

Lemma 2.10. *Let $J \subseteq \{1, \dots, d\}$ with $\#J \geq 1$, and let $x \in B_A$ such that $x_{[q]} = \alpha$ for all $q \in J$. There exists a $\lambda \in \Lambda_x$ with the property: for all $p = 1, \dots, \#J$ there is a $q \in J$ such that $0 < x(\lambda, p)_{[q]} < \alpha$.*

Proof. Using induction on $k \in \mathbb{N}^+$ with $k \leq \#J$ as well as Lemma 2.9 we get a sequence $\lambda = (b_1, \dots, b_{\deg x}) \in \Lambda_x$ with the property: for all $p = 1, \dots, k$ there is a $q \in J$ such that $0 < x(\lambda, p)_{[q]} < \alpha$. In case that $x(\lambda, k)_{[q]} = \alpha$ for some $q \in J$ we can use Lemma 2.9 to get a sequence $(g_1, \dots, g_{\deg x(\lambda, k)}) \in \Lambda_{x(\lambda, k)}$ with $0 < (x(\lambda, k) - g_1)_{[q]} < \alpha$, since $x(\lambda, k) \in B_A$ by Lemma 2.4. By construction it follows that

$$\lambda' = (b_1, \dots, b_k, g_1, \dots, g_{\deg x(\lambda, k)}) \in \Lambda_x,$$

with the property: for all $p = 1, \dots, k+1$ there is a $q \in J$ such that $0 < x(\lambda', p)_{[q]} < \alpha$. Assume that $x(\lambda, k)_{[q]} < \alpha$ for all $q \in J$. In this case λ has already the claimed property. Fix a $p \in \{1, \dots, \#J\}$; we need to show that there is a $q \in J$ with $x(\lambda, p)_{[q]} > 0$ and we are done. Suppose to the contrary that $x(\lambda, p)_{[q]} = 0$ for all $q \in J$. Since $\deg x(\lambda, p) \leq \deg x - \#J$ we get $p = \#J$ by Remark 2.3. Again by Remark 2.3 it follows that $x(\lambda, \#J) = x - (\sum_{q \in J} e_q)$ which contradicts $x \in B_A$, since $x(\lambda, \#J) \in B$. \square

3. THE SEMINORMAL CASE

There are two closely related definitions:

Definition 3.1. Let U be an affine semigroup.

- (1) We call U *normal* if $x \in G(U)$ and $tx \in U$ for some $t \in \mathbb{N}^+$ implies that $x \in U$.
- (2) We call U *seminormal* if $x \in G(U)$ and $2x, 3x \in U$ implies that $x \in U$.

A domain S is called *seminormal* if for every element x in the quotient field $Q(S)$ of S such that $x^2, x^3 \in S$ it follows that $x \in S$. Note that the ring $K[U]$ is seminormal if and only if U is seminormal. This was first observed by Hochster and Roberts in [HR76, Proposition 5.32], provided that $U \subseteq \mathbb{N}^d$. For a proof in the general affine semigroup case we refer to [BG09, Theorem 4.76]. A similar result holds in the normal case, see [Hoc72, Proposition 1] and [BG09, Theorem 4.40]. To get new bounds for the regularity of $K[B]$, we need another characterization. We define the set $\text{Box}(B) := \{x \in B \mid x_{[i]} \leq \alpha \ \forall i = 1, \dots, d\}$.

Theorem 3.2 ([Li04, Theorem 4.1.1]). *The simplicial affine semigroup B is seminormal if and only if B_A is contained in $\text{Box}(B)$.*

In case that $\Gamma_t \subseteq \text{Box}(B)$ for some $t \in \{1, \dots, f\}$ we get $((x - h_t)/\alpha)_{[i]} \in \{0, 1\}$ for all $x \in \Gamma_t$ and for all $i = 1, \dots, d$. Thus, I_t is a squarefree monomial ideal in T if $\Gamma_t \subseteq \text{Box}(B)$. This shows that all ideals in the decomposition are squarefree in the seminormal case.

Lemma 3.3. *Let $\Gamma_t \subseteq \text{Box}(B)$ for some $t \in \{1, \dots, f\}$ with $\Gamma_t \neq \{0\}$. Let $x, y \in \Gamma_t$, and let $i \in \mathbb{N}$ with $1 \leq i \leq d$. We have*

- (1) *If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.*
- (2) *If $0 < x_{[i]} < \alpha$, then $x_{[i]} = y_{[i]}$.*
- (3) *If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} \in \{0, \alpha\}$ and $y_{[i]} = \alpha - x_{[i]}$.*
- (4) *We have $0 < x_{[j]} < \alpha$ and $0 < x_{[k]} < \alpha$ for some $j, k \in \{1, \dots, d\}$ with $j \neq k$.*
- (5) *If $h_{t[i]} > 0$, then $h_{t[i]} = x_{[i]}$.*

Proof. (1) We have $x_{[i]} - y_{[i]} \in \alpha\mathbb{Z}$ and $x_{[i]} - y_{[i]} \in [-\alpha, \alpha]$, since $0 \leq x_{[i]}, y_{[i]} \leq \alpha$. Hence $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.

(2) We have $x_{[i]} - y_{[i]} \notin \{-\alpha, \alpha\}$ and therefore $x_{[i]} = y_{[i]}$ by claim (1).

(3) By claim (1) and (2) we have $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$ and $x_{[i]} \in \{0, \alpha\}$. Hence $y_{[i]} = \alpha - x_{[i]}$.

(4) Suppose that $0 < x_{[j]} < \alpha$ for exactly one $j \in \{1, \dots, d\}$, that is, $x_{[l]} \in \{0, \alpha\}$ for all $l \in \{1, \dots, d\} \setminus \{j\}$. Hence $\sum_{l=1}^d x_{[l]} \notin \alpha\mathbb{N}$ which contradicts $x \in B$. If $x_{[l]} \in \{0, \alpha\}$ for all $l = 1, \dots, d$ we have $x \sim 0$. Hence $0 \in \Gamma_t$, that is, $\Gamma_t = \{0\}$ which contradicts our assumption.

(5) We have $0 < h_{t[i]} \leq x_{[i]} \leq \alpha$ and therefore $h_{t[i]} = x_{[i]}$, since $h_{t[i]} - x_{[i]} \in \alpha\mathbb{Z}$. \square

Proposition 3.4 ([LR10, Theorem 2.2]). *Let B be seminormal. If $\dim K[B] \leq 3$, then the ring $K[B]$ is Cohen-Macaulay.*

Proof. By [Sta78, Theorem 6.4] we need to show that $\#\Gamma_t = 1$ for all $t = 1, \dots, f$. We have $B_A \subseteq \text{Box}(B)$ by Theorem 3.2. The case $d = 2$ follows from Lemma 3.3 (4) and (2). Let $d = 3$; suppose to the contrary that $\#\Gamma_t \geq 2$ for some $t \in \{1, \dots, f\}$. Let $x, y \in \Gamma_t$ with $x \neq y$. We get $0 < x_{[j]} = y_{[j]} < \alpha$ and $0 < x_{[k]} = y_{[k]} < \alpha$ for some $j, k \in \{1, 2, 3\}$ with $j \neq k$ by Lemma 3.3 (4) and (2). By Lemma 3.3 (3) we may assume that $x_{[l]} = \alpha$ and $y_{[l]} = 0$ for $l \in \{1, 2, 3\} \setminus \{j, k\}$. Hence $x - e_l = y \in B$ which contradicts $x \in B_A$. \square

Example 3.5. Proposition 3.4 does not hold for $\dim K[B] > 3$. Consider the monoid

$$B = \langle e_1, \dots, e_4, (1, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 1) \rangle \subset \mathbb{N}^4,$$

with $\alpha = 2$. We have $B_A \subseteq \text{Box}(B)$, thus, B is seminormal by Theorem 3.2. One can show that $(0, 1, 1, 0) + e_1, (0, 1, 1, 0) + e_4 \in B$, but $(0, 1, 1, 0) + e_3 = (0, 1, 3, 0) \notin B$. Hence $K[B]$ is not Buchsbaum by [Tru83, Lemma 3]. Let U be a seminormal positive affine semigroup. Note that $K[U]$ is Cohen-Macaulay if $K[U]$ is Buchsbaum by [BLR06, Proposition 4.15].

Remark 3.6. Consider an element $x \in \text{Box}(B) \cap B_A$. Since $x_{[i]} \leq \alpha$ for all $i = 1, \dots, d$ we have $\deg x \leq d$. On the other hand there is only one element in $\text{Box}(B)$ with degree d , that is, (α, \dots, α) , but $(\alpha, \dots, \alpha) \notin B_A$. This shows that $\deg x \leq d - 1$. By Theorem 3.2 we get $\text{r}(K[B]) \leq d - 1$ if B is seminormal. In Theorem 3.14 we obtain a similar bound for the regularity of $K[B]$ in the seminormal case.

Definition 3.7. For a monomial $m = y_1^{c_1} \cdots y_d^{c_d}$ in T we define $\deg m = \sum_{j=1}^d c_j$. Let I be a monomial ideal in T with minimal set of monomial generators $\{m_1, \dots, m_s\}$. Let $F = y_1^{b_1} \cdots y_d^{b_d}$ be the least common multiple of $\{m_1, \dots, m_s\}$. We define $\text{var}(I) := \deg F$, moreover, we define the set $\text{supp}(I) \subseteq \{1, \dots, d\}$ by $i \in \text{supp}(I)$ if $b_i \neq 0$.

Remark 3.8. Let $t \in \{1, \dots, f\}$; we note that $\tilde{\Gamma}_t$ is always a minimal set of monomial generators of I_t . Moreover, every monomial ideal in T has a unique minimal set of monomial generators. By construction we get that I_t is a proper ideal in T if and only if $\#\Gamma_t \geq 2$. Since $\text{ht } I_t \geq 2$ we have $\text{var}(I_t) \neq 1$. Hence I_t is a proper ideal if and only if $\text{var}(I_t) \geq 2$. Moreover, if I_t is a proper ideal, then $\deg h_t \geq 1$, since $h_t \in G(B) \cap \mathbb{N}^d$ and $h_t \neq 0$.

Consider the squarefree monomial ideal $I = (y_1 y_2, y_2 y_5 y_6)T$ in $T = K[y_1, \dots, y_6]$. We have $\text{var}(I) = 4$ and $\text{supp}(I) = \{1, 2, 5, 6\}$. So $\text{supp}(I)$ is the set of indices of the variables which occur in the minimal generators of a monomial ideal I in T . Note that we always have $\text{var}(I) = \#\text{supp}(I)$ in case that I is a squarefree monomial ideal. Hence $\text{var}(I_t) = \#\text{supp}(I_t)$ if $\Gamma_t \subseteq \text{Box}(B)$ for some $t \in \{1, \dots, f\}$.

Lemma 3.9. *Let $\Gamma_t \subseteq \text{Box}(B)$ for some $t \in \{1, \dots, f\}$. Then $\text{var}(I_t) \leq d - 1 - \deg h_t$.*

Proof. Let $\#\Gamma_t = 1$; we get $\text{var}(I_t) = 0$ and $\deg h_t \leq d - 1$ by Remark 3.6. So we may assume that $\#\Gamma_t \geq 2$. Let $x \in \Gamma_t$; by Lemma 3.3 (4) there are some $j, k \in \{1, \dots, d\}$ with $j \neq k$ such that $0 < x_{[j]}, x_{[k]} < \alpha$. Hence $0 < h_{t[j]}, h_{t[k]} < \alpha$, since $x - h_t \in A$. By Lemma 3.3 (5) we get that $h_{t[q]} = 0$ for all $q \in \text{supp}(I_t)$. We have $\#\text{supp}(I_t) = \text{var}(I_t)$, since I_t is squarefree. Let $J := \{1, \dots, d\} \setminus \text{supp}(I_t)$; we get $j, k \in J$ and $h_{t[q]} \leq \alpha$ for all $q \in J$ it follows that

$$\deg h_t = \frac{1}{\alpha} \sum_{q \in J} h_{t[q]} < d - \#\text{supp}(I_t) = d - \text{var}(I_t).$$

□

Remark 3.10. Consider a normal homogeneous affine semigroup U . One can show that $\text{reg } K[U] \leq \dim K[U] - 1$. This can be deduced from the proof of [HT86, Corollary 4.7] and [HT86, Corollary 3.8], and the fact that $K[U]$ is Cohen-Macaulay by [Hoc72, Theorem 1] or [BG09, Theorem 6.10]. The next Theorem obtains a similar bound for seminormal simplicial affine semigroup rings.

To get new bounds for the regularity of $K[B]$ we need a general bound for the regularity of a monomial ideal. The following is due to Hoa and Trung:

Theorem 3.11 ([HT98, Theorem 3.1]). *Let I be a proper monomial ideal in T . Then*

$$\text{reg } I \leq \text{var}(I) - \text{ht } I + 1.$$

Definition 3.12. We define the set $\Gamma(B) \subseteq \{\Gamma_1, \dots, \Gamma_f\}$ by $\Gamma_t \in \Gamma(B)$ for $t \in \{1, \dots, f\}$ if $\text{reg } K[B] = \text{reg } I_t + \deg h_t$.

By Equation (2.1) we obtain $\Gamma(B) \neq \emptyset$. Note that the ideals and shifts corresponding to the elements of $\Gamma(B)$ are computed by the function `regularityMA` in [BEN].

Proposition 3.13. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \dots, f\}$. If $\Gamma_t \subseteq \text{Box}(B)$, then*

$$\text{reg } K[B] \leq \dim K[B] - 1.$$

Proof. We need to show that $\text{reg } I_t + \deg h_t \leq d - 1$. In case that $\#\Gamma_t = 1$ this follows from Remark 3.6. Assume that $\#\Gamma_t \geq 2$; by Lemma 3.9 and Theorem 3.11 we get

$$(3.1) \quad \text{reg } I_t \leq \text{var}(I_t) - \text{ht } I_t + 1 \leq d - 1 - \deg h_t - 2 + 1 = d - 2 - \deg h_t,$$

since $\text{ht } I_t \geq 2$. Hence $\text{reg } I_t + \deg h_t \stackrel{(3.1)}{\leq} d - 2$ and we are done. □

By Theorem 3.2 and Proposition 3.13 we get the following theorem:

Theorem 3.14. *If B is seminormal, then*

$$\text{reg } K[B] \leq \dim K[B] - 1.$$

Note that the bound established in Theorem 3.14 is sharp. Assume $\alpha \geq d$ in Theorem 4.2; we get $\text{reg } K[B_{d,\alpha}] = d - 1$ and of course $B_{d,\alpha}$ is seminormal. Consider the monoid $B = \langle (3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (1, 0, 2), (0, 2, 1), (1, 1, 1) \rangle$. One can show that $\Gamma_t = \{(2, 2, 2)\}$ for some t and therefore $\Gamma_t \subseteq \text{Box}(B)$. Using MACAULAY2 [GS] we get $\text{reg } K[B] = 2$, hence $\Gamma_t \in \Gamma(B)$. Moreover, since $(4, 2, 0) \in B_A$ it follows that $K[B]$ is not seminormal by Theorem 3.2. Thus, the condition in Proposition 3.13 is not equivalent to B being seminormal.

Proposition 3.15. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \dots, f\}$. If $\Gamma_t \subseteq \text{Box}(B)$ and $\dim K[B] \leq 5$, then*

$$\text{reg } K[B] = r(K[B]).$$

Proof. We have $r(K[B]) \leq \text{reg } K[B]$ by Equation (2.1). We show that $\text{reg } I_t$ is equal to the maximal degree of a generator of I_t . By this we get

$$\text{reg } K[B] = \text{reg } I_t + \deg h_t = \max \{ \deg x \mid x \in \Gamma_t \},$$

and hence $r(K[B]) \geq \text{reg } K[B]$. Keep in mind that I_t is squarefree. The case $\#\Gamma_t = 1$ follows from construction. We therefore may assume that $\#\Gamma_t \geq 2$, or equivalently, $\text{var}(I_t) \geq 2$; note that $\deg h_t \geq 1$, see Remark 3.8. Let $d \leq 3$; by Lemma 3.9 we get $\text{var}(I_t) \leq 1$ which contradicts $\#\Gamma_t \geq 2$. Let $d = 5$; by Lemma 3.9 we have to consider the cases $\text{var}(I_t) \in \{2, 3\}$. Let $\text{var}(I_t) = 2$; the ideal I_t is of the form $I_t = (y_k, y_l)T$ for some $k, l \in \{1, \dots, 5\}$ with $k \neq l$, since $\text{ht } I_t \geq 2$. It follows that $\text{reg } I_t = 1$. By a similar argument we get the assertion for $d = 4$ and $\text{var}(I_t) = 2$. Let $d = 5$ and $\text{var}(I_t) = 3$. Since $\text{ht } I_t \geq 2$ the only ideals possible are

$$I_{t_1} = (y_k, y_l, y_m)T, I_{t_2} = (y_k y_l, y_m)T, I_{t_3} = (y_k y_l, y_k y_m, y_l y_m)T$$

for some $k, l, m \in \{1, \dots, 5\}$ which are pairwise not equal. By Theorem 3.11 we get $\text{reg } I_{t_1} = 1$ and $\text{reg } I_{t_2} = \text{reg } I_{t_3} = 2$ and we are done. \square

By Theorem 3.2 and Proposition 3.15 it follows that $\text{reg } K[B] = r(K[B])$ if B is seminormal and $\dim K[B] \leq 5$. Thus, the Eisenbud-Goto conjecture holds in this case by Proposition 2.7. Theorem 3.18 will confirm the conjecture in any dimension in the seminormal case. Note that Proposition 3.15 could fail for $d \geq 6$. Let us consider the squarefree monomial ideal $I = (y_1 y_2, y_3 y_4)T$ with $\text{var}(I) = 4$. So $\text{reg } I = 3$ is bigger than the maximal degree of a generator of I which is 2.

Lemma 3.16. *Let $\Gamma_t \subseteq \text{Box}(B)$ for some $t \in \{1, \dots, f\}$. Let $n \in \Gamma_t$ and $m \in \tilde{\Gamma}_t$ such that $m = y^{(n-h_t)/\alpha}$. Then*

- (1) $n_{[q]} = 0$ for all $q \in \text{supp}(I_t) \setminus \text{supp}(mT)$.
- (2) $n_{[q]} = \alpha$ for all $q \in \text{supp}(mT)$.

Proof. (1) Suppose to the contrary that there is a $q \in (\text{supp}(I_t) \setminus \text{supp}(mT)) \neq \emptyset$ such that $n_{[q]} > 0$. Since $q \in \text{supp}(I_t)$ we have $h_{t[q]} = 0$ by Lemma 3.3 (5), and therefore $n_{[q]} = \alpha$, since $h_{t[q]} - n_{[q]} \in \alpha\mathbb{Z}$ and $n_{[q]} \leq \alpha$. This implies $q \in \text{supp}(mT)$ which is a contradiction.

(2) Let $q \in \text{supp}(mT)$; we have $n_{[q]} \geq \alpha$. Moreover, we get $n_{[q]} \leq \alpha$, since $\Gamma_t \subseteq \text{Box}(B)$. \square

The above Lemma is false in general. For the affine semigroup B in Example 2.8 we have $\Gamma_4 = \{(6, 2), (2, 6)\}$, that is, $h_4 = (2, 2)$, and $\tilde{\Gamma}_4 = \{y_1, y_2\}$. For $n \in \Gamma_4$ we get that $n_{[i]} > 0$ for $i = 1, 2$. But $\text{supp}(I_4) = \{1, 2\}$ and $\#\text{supp}(y_1 T) = \#\text{supp}(y_2 T) = 1$. As a consequence of the next proposition the Eisenbud-Goto conjecture holds if B is seminormal.

Proposition 3.17. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \dots, f\}$. If $\Gamma_t \subseteq \text{Box}(B)$, then*

$$\text{reg } K[B] \leq \deg K[B] - \text{codim } K[B].$$

Proof. By construction we need to show that $\text{reg } I_t + \deg h_t \leq \deg K[B] - c$. If $\#\Gamma_t = 1$ the assertion follows from Proposition 2.7. Let $\#\Gamma_t \geq 2$, equivalently, I_t is a proper ideal, see Remark 3.8. We have $\Gamma_t = \{n_1, \dots, n_{\#\Gamma_t}\}$ and $\tilde{\Gamma}_t = \{m_1, \dots, m_{\#\Gamma_t}\}$; we may assume that $m_i = y^{(n_i - h_t)/\alpha}$. We set $J_k := (m_1, \dots, m_k)T$ and $g(k) := \text{var}(J_k) - \text{ht } J_k + 1 + \deg h_t$ for $k \in \mathbb{N}$ with $1 \leq k \leq \#\Gamma_t$. Note that $J_{\#\Gamma_t} = I_t$, moreover, J_k is a (proper) squarefree

monomial ideal in T , since $\Gamma_t \subseteq \text{Box}(B)$, hence $\text{var}(J_k) = \#\text{supp}(J_k)$. We show by induction on $k \in \mathbb{N}$ with $1 \leq k \leq \#\Gamma_t$ that there is a set L_k with the following properties

- (i) $L_k \subseteq B_A$.
- (ii) $\#L_k \geq g(k) - 1$.
- (iii) $x \not\sim y$ for all $x, y \in L_k$ with $x \neq y$.
- (iv) $\deg x \geq 2$ for all $x \in L_k$.
- (v) $x_{[q]} = 0$ for all $x \in L_k$ and for all $q \in \text{supp}(I_t) \setminus \text{supp}(J_k)$.

Let $k = 1$. We have $\text{ht } J_1 = 1$ and $\text{var}(J_1) + \deg h_t = \deg n_1$, that is, $g(1) = \deg n_1$. Fix a $\lambda \in \Lambda_{n_1}$ and set

$$L_1 := \{n_1(\lambda, 0), \dots, n_1(\lambda, \deg n_1 - 2)\},$$

clearly $\#L_1 = \deg n_1 - 1 = g(1) - 1$, hence (ii) is satisfied and by construction we get property (iv). By Lemma 2.4 (1) $L_1 \subseteq B_A$ which shows (i), and by Lemma 2.4 (2) property (iii) holds. By Lemma 3.16 (1) we get $n_1(\lambda, 0)_{[q]} = 0$ for all $q \in \text{supp}(I_t) \setminus \text{supp}(J_1)$, hence (v) holds by construction of L_1 .

Using induction on $k \leq \#\Gamma_t - 1$ the properties (i)-(v) hold for L_k . We define the set $J := \text{supp}(m_{k+1}T) \setminus \text{supp}(J_k)$. By Lemma 3.16 (2) we get $n_{k+1}[q] = \alpha$ for all $q \in \text{supp}(m_{k+1}T)$. Since $n_{k+1} \in B_A$ it follows that $\deg n_{k+1} \geq \#\text{supp}(m_{k+1}T) + 1$. Moreover, since $n_{k+1}[q] = \alpha$ for all $q \in J$ we can fix by Lemma 2.10 a $\lambda \in \Lambda_{n_{k+1}}$ with the property: for all $p = 1, \dots, \#J$ there is a $q \in J$ with $0 < n_{k+1}(\lambda, p)_{[q]} < \alpha$. There could be two different cases:

Case 1: $\text{supp}(J_k) \cap \text{supp}(m_{k+1}T) \neq \emptyset$. (e. g., $k = 2$, $J_2 = (y_1y_2, y_2y_3y_4)T$, and $m_3 = y_4y_5y_6$.)
Set

$$L_{k+1} := L_k \cup \{n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \#J)\}.$$

In case that $J = \emptyset$ we set $L_{k+1} := L_k$.

(iii) By induction we get $x \not\sim y$ for all $x, y \in L_k$ with $x \neq y$, moreover, $n_{k+1}(\lambda, i) \not\sim n_{k+1}(\lambda, j)$ for all $i, j \in \mathbb{N}$ with $0 \leq i < j \leq \deg n_{k+1}$ by Lemma 2.4 (2). Fix an $x \in L_k$ and let $p \in \{1, \dots, \#J\}$. By property (v) $x_{[q]} = 0$ for all $q \in J$, moreover, there is a $q \in J$ such that $0 < n_{k+1}(\lambda, p)_{[q]} < \alpha$, hence $x \not\sim n_{k+1}(\lambda, p)$. Thus, property (iii) is satisfied. This also shows that $\#L_{k+1} = \#L_k + \#J$.

(i) By Lemma 2.4 (1) $n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \#J) \in B_A$, since $n_{k+1} \in B_A$.

(iv) Since $\#\text{supp}(m_{k+1}T) \geq \#J + 1$ we obtain $\deg n_{k+1} \geq \#J + 2$. Hence (iv) holds by construction.

(v) By induction $x_{[q]} = 0$ for all $x \in L_k$ and for all $q \in (\text{supp}(I_t) \setminus \text{supp}(J_k)) \supseteq (\text{supp}(I_t) \setminus \text{supp}(J_{k+1}))$. By Lemma 3.16 (1) we have $n_{k+1}[q] = 0$ for all $q \in (\text{supp}(I_t) \setminus \text{supp}(m_{k+1}T)) \supseteq (\text{supp}(I_t) \setminus \text{supp}(J_{k+1}))$, hence property (v) holds by construction.

(ii) Since $\text{supp}(J_{k+1}) = \text{supp}(J_k) \cup \text{supp}(m_{k+1}T)$ we get $\text{var}(J_{k+1}) = \text{var}(J_k) + \#J$. We have $\text{ht } J_{k+1} \geq \text{ht } J_k$ and therefore

$$g(k+1) - 1 \leq \#J + \text{var}(J_k) - \text{ht } J_k + 1 + \deg h_t - 1 = \#J + g(k) - 1 \leq \#J + \#L_k = \#L_{k+1}.$$

Case 2: $\text{supp}(J_k) \cap \text{supp}(m_{k+1}T) = \emptyset$. (e. g., $k = 2$, $J_2 = (y_1y_2, y_2y_3y_4)T$, and $m_3 = y_5y_6y_7$.)
Note that $J = \text{supp}(m_{k+1}T)$, in particular, $\#J \geq 1$. Set

$$L_{k+1} := L_k \cup \{n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \#J - 1)\}.$$

In case that $\#J = 1$ we set $L_{k+1} := L_k$.

(iii), (i), (iv), (v) Analogous, replace $\#J$ by $\#J - 1$ in the corresponding proofs in the first case. Moreover, $\#L_{k+1} = \#L_k + \#J - 1$ by construction.

(ii) We also have $\text{var}(J_{k+1}) = \text{var}(J_k) + \#J$. Since $\text{supp}(J_k) \cap \text{supp}(m_{k+1}T) = \emptyset$ we get that

$m_{k+1} + J_k$ is a non-zero-divisor of T/J_k . Hence $\text{ht } J_{k+1} = \text{ht } J_k + 1$ by Krull's Principal Ideal Theorem, see [Eis95, Theorem 10.1], and therefore

$$g(k+1) - 1 = \#J + \text{var}(J_k) - \text{ht } J_k - 1 + 1 + \deg h_t - 1 = \#J + g(k) - 2 \leq \#J + \#L_k - 1 = \#L_{k+1}.$$

By this we obtain a set $L_{\# \Gamma_t}$ with the above properties, in particular

$$(3.2) \quad \#L_{\# \Gamma_t} \stackrel{(ii)}{\geq} g(\# \Gamma_t) - 1 = \text{var}(I_t) - \text{ht } I_t + 1 + \deg h_t - 1 \geq \text{reg } I_t + \deg h_t - 1,$$

by Theorem 3.11. We get a set

$$L := L_{\# \Gamma_t} \cup \{0, a_1, \dots, a_c\},$$

with $L \subseteq B_A$ such that $x \not\sim y$ for all $x, y \in L$ with $x \neq y$ by (i), (iii), and Remark 2.6. Since $\deg K[B] = f$ (see Proposition 2.1) we have

$$\deg K[B] \geq \#L \stackrel{(iv)}{=} \#L_{\# \Gamma_t} + c + 1 \stackrel{(3.2)}{\geq} \text{reg } I_t + \deg h_t + c = \text{reg } K[B] + c.$$

□

We therefore obtain from Theorem 3.2 and Proposition 3.17 the following main result:

Theorem 3.18. *If B is seminormal, then*

$$\text{reg } K[B] \leq \deg K[B] - \text{codim } K[B].$$

Note that the bound of Theorem 3.18 is again sharp. For $d = 2$ and $\alpha \geq 2$ we get that $\text{reg } K[B_{2,\alpha}] = \lfloor 2 - \frac{2}{\alpha} \rfloor = 1$ and $\deg K[B_{2,\alpha}] - \text{codim } K[B_{2,\alpha}] = \alpha - (\alpha + 1) + 2 = 1$, see Section 4.

4. REGULARITY OF FULL VERONESE RINGS

For $X, Y \subseteq \mathbb{N}^d$ we define $X + Y := \{x + y \mid x \in X, y \in Y\}$, $nX := X + \dots + X$ (n -times), and $0X := 0$. Recall that $M_{d,\alpha} = \{(u_1, \dots, u_d) \in \mathbb{N}^d \mid \sum_{i=1}^d u_i = \alpha\}$ and $B_{d,\alpha}$ denotes the submonoid of $(\mathbb{N}^d, +)$ which is generated by $M_{d,\alpha}$. For example $B_{2,2} = \langle (2, 0), (0, 2), (1, 1) \rangle$. We have

$$(4.1) \quad nM_{d,\alpha} = \left\{ (u_1, \dots, u_d) \in \mathbb{N}^d \mid \sum_{i=1}^d u_i = n\alpha \right\},$$

hence there is an isomorphism of K -vector spaces: $K[B_{d,1}]_{n\alpha} \cong K[B_{d,\alpha}]_n$. It is a well known fact that $h_{K[B_{d,1}]}(n) = \binom{n+d-1}{d-1}$, where h_M denotes the Hilbert polynomial. This shows that $h_{K[B_{d,\alpha}]}(n) = h_{K[B_{d,1}]}(n\alpha) = \binom{n\alpha+d-1}{d-1}$ and therefore $\deg K[B_{d,\alpha}] = \alpha^{d-1}$. Moreover, we get $\text{codim } K[B_{d,\alpha}] = \binom{\alpha+d-1}{d-1} - d$, since $\#M_{d,\alpha} = \binom{\alpha+d-1}{d-1}$. The semigroups $B_{d,\alpha}$ are normal, hence the ring $K[B_{d,\alpha}]$ is Cohen-Macaulay by [Hoc72, Theorem 1] and therefore $\# \Gamma_t = 1$ for all $t = 1, \dots, f$, see [Sta78, Theorem 6.4]. It follows that

$$(4.2) \quad \text{reg } K[B_{d,\alpha}] = r(K[B_{d,\alpha}]),$$

by Equation (2.1). In the following we will compute the reduction number $r(K[B_{d,\alpha}])$ which can also be computed by $r(K[B_{d,\alpha}]) = \min \{r \in \mathbb{N} \mid rM_{d,\alpha} + \{e_1, \dots, e_d\} = (r+1)M_{d,\alpha}\}$, see [HS03, Page 129,135].

Lemma 4.1. *Let $r \in \mathbb{N}$. The following assertions are equivalent:*

- (1) $rM_{d,\alpha} + \{e_1, \dots, e_d\} = (r+1)M_{d,\alpha}$.
- (2) $(r+1)\alpha > d(\alpha-1)$.

Proof. (1) \Rightarrow (2) Assume that $0 \leq (r+1)\alpha \leq d(\alpha-1)$. There is an element $x \in \mathbb{N}^d$ with $x_{[j]} \leq \alpha-1$ for all $j = 1, \dots, d$ and $\sum_{j=1}^d x_{[j]} = (r+1)\alpha$. We have $x \in (r+1)M_{d,\alpha}$ by Equation (4.1). Suppose that $x \in rM_{d,\alpha} + \{e_1, \dots, e_d\}$ we get $x = x' + e_j$ for some $x' \in \mathbb{N}^d$ and some $j \in \{1, \dots, d\}$ which contradicts $x_{[j]} \leq \alpha-1$. Hence $x \notin rM_{d,\alpha} + \{e_1, \dots, e_d\}$.
 (2) \Rightarrow (1) Let $x \in (r+1)M_{d,\alpha}$. Suppose that $x_{[j]} \leq \alpha-1$ for all $j = 1, \dots, d$. We get $(r+1)\alpha = \sum_{j=1}^d x_{[j]} \leq d(\alpha-1)$. Thus, $x_{[j]} \geq \alpha$ for some $j \in \{1, \dots, d\}$ and therefore $x - e_j \in rM_{d,\alpha}$ by Equation (4.1). Hence $(r+1)M_{d,\alpha} \subseteq rM_{d,\alpha} + \{e_1, \dots, e_d\}$, that is, $(r+1)M_{d,\alpha} = rM_{d,\alpha} + \{e_1, \dots, e_d\}$ and we are done. \square

Theorem 4.2. *We have*

$$\operatorname{reg} K[B_{d,\alpha}] = \lfloor d - \frac{d}{\alpha} \rfloor.$$

Proof. By Equation (4.2) we need to show that $\operatorname{r}(K[B_{d,\alpha}]) = \lfloor d - \frac{d}{\alpha} \rfloor$. We have

$$(\lfloor d - \frac{d}{\alpha} \rfloor + 1) \alpha > (d - \frac{d}{\alpha}) \alpha = d(\alpha - 1),$$

hence $\operatorname{r}(K[B_{d,\alpha}]) \leq \lfloor d - \frac{d}{\alpha} \rfloor$ by Lemma 4.1. We may assume that $\lfloor d - \frac{d}{\alpha} \rfloor \geq 1$. We get

$$(\lfloor d - \frac{d}{\alpha} \rfloor - 1 + 1) \alpha \leq (d - \frac{d}{\alpha}) \alpha = d(\alpha - 1),$$

hence $\operatorname{r}(K[B_{d,\alpha}]) > \lfloor d - \frac{d}{\alpha} \rfloor - 1$ by Lemma 4.1 and we are done. \square

Example 4.3. By Theorem 4.2 we are able to compute the Castelnuovo-Mumford regularity of full Veronese rings. For $B_{20,2}$ we know that $\operatorname{reg} K[B_{20,2}] = \lfloor 20 - \frac{20}{2} \rfloor = 10$. Moreover, we have $\deg K[B_{20,2}] - \operatorname{codim} K[B_{20,2}] = 2^{19} - \binom{2+19}{19} + 20 = 524098$.

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